Correspondence between QCD sum rules and constituent quark models

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Abstract. We compare two widely used approaches to the description of hadron properties: QCD sum rules and constituent quark models. Making use of the dispersion formulation of the quark model, we show that both approaches lead to similar spectral representations for the hadron observables with the important difference that the quark model is based on Feynman diagrams with massive quarks, whereas QCD sum rules are based on the same Feynman diagrams for current quarks with the additional condensate contributions for light quarks and gluons. We give arguments for the similarity of the smearing function in the sum rule case and the hadron wave function of the quark model. Analyzing the sum rule for the leptonic decay constant of the heavy pseudoscalar meson containing a light u or s quark, we find that the quark condensates at the chiral symmetry-breaking scale $\mu_\chi \simeq 1\,\text{GeV}$, $\langle \bar{u}u \rangle = -(230 \pm 15\,\text{MeV})^3$ and $\langle \bar{s}s \rangle = -(220 \pm 15\,\text{MeV})^3$ correspond to constituent quark masses of $m_u \simeq 220\,\text{MeV}$ and $m_s \simeq 350\,\text{MeV}$, respectively. We also obtain the running of the quark-model parameters above the chiral scale μ_χ. The observed correspondence between constituent quark models and QCD sum rules allows for a deeper understanding of both methods and their parameters. It also provides a QCD basis for constituent quark models, extending their applicability above the scale of chiral symmetry breaking.

1 Introduction

There are several pieces of evidence that the static properties of hadrons and their characteristics in processes with momentum transfers not larger than a few GeV may be well described treating the hadron as a relativistic few-body composite system of effective particles: the constituent quarks. This evidence comes from several sources. Among them are the following.

(i) *Hadron spectroscopy*, where mesons and baryons spectra may be well described in the relativistic constituent quark model [1];

(ii) *high-energy hadron–hadron and hadron–nucleus scattering* at small and intermediate momentum transfers, where we have momentum distributions of secondary particles; this is in fact a scattering of the three-quark and two-quark bound states, well described assuming that mesons and nucleons are bound states of two and three constituent quarks, respectively [2];

(iii) *photon–hadron scattering at small momentum transfers*, where the observables speak in favor of the presence of a few extended objects inside hadrons [3];

(iv) *exclusive processes at small and intermediate momentum transfers*, where the constituent quark picture has

been successfully applied to the calculation of elastic and transition form factors. Indeed, various models based on the notion of constituent quarks can be found in the literature, for instance the dispersion approach [4,5], the quasipotential approach [6], and light-front [7,8], instant-form [9] and point-form [10] quark models. For more details we refer to the review in [11]. The constituent quark masses are the parameters to be adjusted by describing the data. The typical values of the constituent masses are $m_u = m_d \simeq 220 \div 300\,\text{MeV}$, $m_s \simeq 350 \div 450\,\text{MeV}$, $m_c \simeq 1.4 \div 1.6\,\text{GeV}$ and $m_b \simeq 4.8 \div 5.0\,\text{GeV}$.

The many successes of the constituent quark model to describe the data definitely prompt us to accept that this approach provides a relevant description of the *non-perturbative* QCD physics at low and intermediate momentum transfers. However, on one hand a rigorous derivation of the quark model from the QCD Lagrangian is hard to establish. On the other hand a relationship between QCD Lagrangian and hadron physics is provided by QCD sum rules. Thus it is reasonable to look for a correspondence between quark models and sum rules in order to connect the former to QCD in this way.

In this paper we demonstrate such a correspondence between sum rules and quark models making use of the dispersion formulation for the latter [12]. This correspondence not only provides a QCD basis for quark-model

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calculations but also helps in obtaining a better under-
standing of the different versions of QCD sum rules and
of the parameters which a priori have no clear physical
meaning within QCD sum rules. Moreover, it allows us to
understand the proper running of the constituent quark-
model parameters above the chiral symmetry-breaking
scale $\mu_\chi \simeq 1\,\text{GeV}$, opening the possibility to apply con-
stituent quark models above this scale in a controlled way.

1.1 QCD sum rules

The method of QCD sum rules [13] is based on the follow-
ing theoretical concepts: a complicated structure of the
physical QCD vacuum, operator product expansion and
quark–hadron duality. The first concept states that the
physical QCD vacuum is different from the perturbative
QCD vacuum, and properties of the former may be de-
scribed in terms of the condensates, i.e. non-vanishing am-
plitudes of local gauge-invariant operators over the phys-
ical vacuum. Perturbative QCD calculations can still be
applied far from hadronic thresholds, but require modifi-
cations: non-perturbative contributions given by the con-
densates appear as power corrections to the usual per-
turbative expressions. These proper modifications of the
perturbation theory may be obtained using the operator
product expansion.

The central object considered within the QCD sum
rule method is the correlator of the quark currents over the
physical vacuum. One obtains spectral representations for
this correlator by two different means: within the modified
QCD perturbation theory including condensates, and us-
ing hadron saturation. Local quark–hadron duality states
that both representations for the spectral density should
be equal to each other after a proper smearing (the same
for the QCD part and the hadronic part) is applied. The
two smeared representations for the spectral density give
the two sides of the QCD sum rule.

Sum rules as a technical tool to obtain the resonance
properties from QCD is based on attempting to choose
the smearing function such that a single resonance domi-
nates the hadronic part of the sum rule, and at the same
time only a few condensates of the lowest dimension are
essential on the QCD side. In many cases it is possible
to find smearing functions which satisfy the above com-
peting requirements [14]. Then one obtains the resonance
parameters, such as masses, decay constants, or form fac-
tors. For details we refer to the review papers in [15,16]
and references therein.

In practice, the application of this method leads to
the calculation of the spectral densities of the relevant
Feynman diagrams with current quarks and gluons, taking
the convolution of this spectral density with the smearing
function, and adding non-perturbative power corrections
described in terms of the condensates. Depending on the
choice of the smearing function one obtains various ver-
sions of the sum rules (moment, Borel, Gaussian, etc.).
The resulting sum rules contain physical parameters such
as quark masses and condensates, and parameters describ-
ing the details of the smearing procedure which may vary

from one observable to the other. They are fixed by re-
quiring the stability of the sum rule or from fits to the
data.

1.2 Constituent quarks and the dispersion approach

The dispersion approach [12] uses the constituent quarks
and is thus conceptually quite different. Nevertheless,
technically it is based on calculating the spectral densities
of the *same diagrams* as in the sum rules, but involving
the constituent quarks, and taking the convolution of these
spectral densities with the wave functions of the partici-
pating hadrons. All non-perturbative effects are assumed
to be taken into account by introducing the constituent
quarks and no other non-perturbative contributions are
added. One may treat the wave functions either as some
non-perturbative input and use simple parameterizations
for them, or use the relativistic wave functions obtained
from the solutions to an eigenvalue problem [8].

The central observation for comparing sum rules and
quark models is that the densities in the spectral represen-
tations are the same functions in both methods. The dif-
ferences come from the following sources: in quark models
one uses effective constituent quark masses and the wave
functions of hadrons; in sum rules one uses current quark
masses, smearing functions which may have no physical
meaning, and adds the contributions of condensates. The
spectral densities determine the main qualitative features
of the calculated hadron observables, such as their de-
pendence on the momentum transfer or scaling proper-
ties in the heavy-quark mass. On the quantitative side, if
both approaches give the correct description of the hadron
properties, the constituent masses in quark-model calcu-
lations should numerically reproduce the contribution of
the condensates in QCD sum rules.

We make this correspondence explicit by analyzing the
decay constant of the B meson, a pseudoscalar meson con-
taining heavy b and light $\bar{u}$ quarks. We obtain the con-
stituent mass of the light quark from QCD sum rules in
the limit $m_b \to \infty$.

We argue that the above correspondence between sum
rules and quark model and the knowledge of the spectro-
scopic wave functions in the latter allow one to find the
optimal smearing function and to favor a specific version
of QCD sum rules depending on the hadron considered[1].
For instance, for hadrons containing heavy quarks we give
arguments in favor of Gaussian sum rules compared to
Borel sum rules, while in the case of hadrons containing
light quarks only Borel sum rules appear to be favored.

The correspondence between constituent quark mod-
els and QCD itself (via the sum rules) can be pushed fur-
ther. Indeed the natural upper limit of applicability of
constituent quark models is the scale of chiral symmetry
breaking, $\mu_\chi \simeq 1\,\text{GeV}$. Below such a scale the constituent
mass may be approximated by a constant fixed mainly

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[1] Notice that the idea of a similarity of the Borel wave func-
tion of QCD sum rules and the light-cone wave function of a
hadron in terms of current quarks was discussed in [17].

by spectroscopic data on hadron masses. Above the chiral scale a running of the constituent mass is strongly expected. A similar situation holds as well for the hadron wave functions. The correspondence we found between constituent quark models and QCD sum rules allows us to find out quantitatively the running of both the constituent mass and average momentum inside the hadron just by imposing the appropriate QCD scale dependence of the quark condensate.

This paper is organized as follows: in Sect. 2 the main formulas for the pseudoscalar B meson in the dispersion approach are recalled. In Sect. 3 we consider a sum rule for the decay constant f_B and its relationship to the quark model, and obtain the constituent quark mass. We also discuss physical motivations for the smearing function. In Sect. 4 we address and solve quantitatively the relation between the QCD running of the quark condensate and the running of both the constituent mass and average momentum inside the hadron. Finally we summarize our results in Sect. 5.

2 Dispersion approach based on the constituent quark picture

We present here the main formulas of the dispersion approach [12] necessary for our discussion. The relativistic wave function of the B meson consisting of b and $\bar{u}$ quarks with masses m_b and m_u, is normalized as

$$1 = \int\limits_{(m_b+m_u)^2}^{\infty} \mathrm{d}s \, |\psi(s)|^2 \, \rho(s, m_b^2, m_u^2), \quad (1)$$

where $\rho(s)$ is the spectral density of the Feynman loop graph with the $\mathrm{i}\gamma_5$ Dirac structures in the vertices (see Fig. 1), given explicitly by
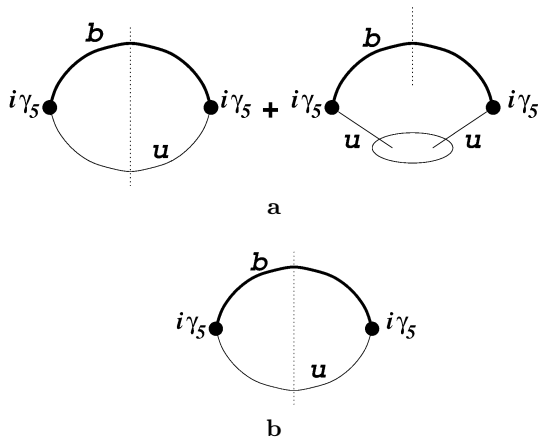

Fig. 1. The cut Feynman diagrams for calculating the spectral density $\rho(s)$ of the correlator (6). The dotted line denotes the cut. **a** The perturbative loop with b quark and the current u quark and the contribution of the quark condensate. **b** The loop diagram containing the b quark and the constituent u quark with the mass m_u

$$\rho(s, m_b^2, m_u^2)$$
$$= -\frac{N_c}{8\pi^3} \int \mathrm{d}k_u \mathrm{d}k_b \delta(m_b^2 - k_b^2)\delta(m_u^2 - k_u^2)\delta(\tilde{p} - k_b - k_u)$$
$$\times \mathrm{Sp}\left((m_b + \hat{k}_b)\mathrm{i}\gamma_5(m_u - \hat{k}_u)\mathrm{i}\gamma_5\right)$$
$$= \frac{N_c}{8\pi^2} \frac{\lambda^{1/2}(s, m_b^2, m_u^2)}{s}\left(s - (m_b - m_u)^2\right)$$
$$\times\theta\left(s - (m_b + m_u)^2\right). \quad (2)$$

Here $s = \tilde{p}^2$ and $\lambda(s, m_b^2, m_u^2) \equiv (s - m_b^2 - m_u^2)^2 - 4m_b^2 m_u^2$. The normalization condition (1) corresponds to the normalization of the charged form factor of the meson at $q^2 = 0$.

The elastic form factor describing the interaction of the b quark with an external vector field is defined as

$$\langle B(p')|\bar{b}\gamma_\mu b|B(p)\rangle = (p + p')_\mu F(q^2), \quad (3)$$

with $p - p' = q$. For $q^2 < 0$ one finds

$$F(q^2) = \int \mathrm{d}s\mathrm{d}s'\psi(s)\psi(s')\Delta_V(s', s, q^2|m_b^2, m_b^2, m_u^2), \quad (4)$$

where $\Delta_V(s', s, q^2|m_b^2, m_b^2, m_u^2)$ is the double spectral density of the triangle diagram, whose explicit expression can be found in [12]. An important property of the spectral densities is the relation

$$\Delta_V(s', s, q^2|m_b^2, m_b^2, m_u^2) \to \delta(s - s')\rho(s, m_b^2, m_u^2), \quad (5)$$

valid for $q^2 \to 0$. For a given wave function $\psi(s)$ (4) allows us to calculate the form factor for $q^2 \le 0$. The relation (5) guarantees that $F(q^2 = 0) = 1$.

3 QCD sum rules and the constituent quark mass

We now turn to the calculation of resonance observables in QCD sum rules and start with the decay constant f_B. The central object in this case is the correlator of two pseudoscalar currents

$$\mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}qx}\langle\mathrm{vac}|T\{\bar{b}(x)\gamma_5 u(x)\bar{u}(0)\gamma_5 b(0)\}|\mathrm{vac}\rangle. \quad (6)$$

Here $|\mathrm{vac}\rangle$ denotes the physical QCD vacuum which differs from the perturbative QCD vacuum. The physical QCD vacuum is characterized by non-vanishing expectation values of gauge-invariant operators. These expectation values vanish in the perturbation theory (i.e. when averaging over the perturbative vacuum state). One can write the correlator (6) as the dispersion representation in q^2 and calculate the spectral density of this representation using either the language of hadronic intermediate states or QCD intermediate states. Clearly the two spectral densities look quite different when compared point-by-point. The local quark–hadron duality states that both spectral densities are still equal to each other after a proper smearing is applied to both of them. For more details we refer to

[15]. The perturbative part of the QCD spectral density contains the loop diagram and the radiative corrections to it. The non-perturbative part of the QCD spectral density contains contributions of the condensates. It is known that the radiative corrections play a role for obtaining realistic predictions (see [18,19] for details and references). However, since radiative corrections are not essential for the (non-perturbative) correspondence we are looking for, we postpone its discussion to the next section and keep on the QCD side only the contribution of the loop diagram and of the quark condensate. On the hadronic side we keep only the B meson contribution and assume that the hadron continuum may be suppressed or effectively taken into account by the choice of the smearing function. The resulting sum rule takes the form

$$\frac{f_B^2 M_B^4}{(\overline{m}_b + \overline{m}_u)^2} H(M_B^2) \tag{7}$$

$$= \int ds\, \rho(s, \overline{m}_b^2, \overline{m}_u^2) H(s) - \overline{m}_b \langle \bar{q}q \rangle H\left((\overline{m}_b + \overline{m}_u)^2\right).$$

Here $\overline{m}_u \leq 10\,\text{MeV}$ and $\overline{m}_b$ are the short-distance quark masses.[2] $H(s)$ is the function which provides the relevant smearing of the spectral densities on both hadronic and the QCD parts of the sum rule. Let us now rewrite the sum rule (7) in the form

$$1 = \int ds\, \rho(s, \overline{m}_b^2, \overline{m}_u^2)\, \Phi^2(s) - \overline{m}_b \langle \bar{q}q \rangle \Phi\left((\overline{m}_b + \overline{m}_u)^2\right), \tag{8}$$

where we have defined $\Phi(s)$ as

$$\Phi(s) = \frac{\overline{m}_b + \overline{m}_u}{f_B M_B^2} \sqrt{H(s)/H(M_B^2)}. \tag{9}$$

It is clear that as soon as we describe the B meson by the pseudoscalar interpolating current $\bar{b}i\gamma_5 u$, the combination (9) will appear for any observable as the result of applying smearing in the corresponding channel. Usually, the smearing functions are not assumed to be universal and are allowed to vary from one observable to another. Imagine however that there exists such a smearing function which turns out to be independent of the quantity described by sum rules and is thus universal. Then up to condensate contributions the expression (8) looks like the normalization condition for this wave function. Comparing (8) and (1) we may expect that the smearing function $\Phi(s)$ and the wave function $\Psi(s)$ are close to each other and that the appearance of the constituent quark effectively accounts for the contribution of the condensates.

To make this point clear let us consider a simple theory without condensates but with confinement. A non-relativistic potential model with a confining potential represents an example of such a theory. In this case the spectrum of states is discrete and one can use sum rules to

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[2] If radiative corrections are included in the spectral density of (7), these quantities depend on the scheme and scale. Hereafter we imply the use of the $\overline{\text{MS}}$ renormalization scheme. The inclusion of radiative corrections within the constituent quark picture was discussed in [20].

calculate the bound-state parameters such as their masses and decay constants [14] and to study quark–hadron duality [21]. One can also use sum rules to calculate the transition form factors. The merit of referring to the potential model is that the *exact* solutions for the above quantities are known. So confronting the exact result with the approximate results obtained by sum rules allows one to understand the accuracy of the method and to motivate the choice of the smearing functions. For the resonance masses and the decay constants this was done in [14]. To understand what happens for the form factors, let us study the form factor describing the transition from the state i to the state j, $F_{ij}(\mathbf{q}^2)$. The corresponding expression is known:

$$F_{i \to j}^{\text{exact}}(\mathbf{q}^2) = \int d\mathbf{k}\, \Psi_i(\mathbf{k}^2)\Psi_j^*\left((\mathbf{k} + \mathbf{q})^2\right). \tag{10}$$

Here $\Psi_n(\mathbf{k}^2)$ is the wave function of the state n obtained by solving the Schrödinger equation. Equation (10) can be written in the form of a double dispersion integral

$$F_{i \to j}^{\text{exact}}(\mathbf{q}^2) \tag{11}$$

$$= \int dz dz'\, \Psi_i(z)\Psi_j^*(z')\Delta_{\text{NR}}(z, z', \mathbf{q}^2|m^2),$$

where $\Delta_{\text{NR}}(z, z', \mathbf{q}^2|m^2)$ is the double spectral density of the triangle diagram of the non-relativistic field theory calculated using the non-relativistic Green functions of free quarks [2]. Since condensates are absent in this theory, the application of sum rules would lead to a similar expression:

$$F_{i \to j}^{\text{SR}}(\mathbf{q}^2) \tag{12}$$

$$= \int dz \Phi_i(z) dz' \Phi_j^*(z')\Delta_{\text{NR}}(z, z', \mathbf{q}^2|m^2),$$

where $\Phi_n(z)$ is the smearing function for the state n. Choosing the smearing function $\Phi_n(z)$ equal to the exact function $\Psi_n(z)$ leads to the exact form factor. The same argument applies to other characteristics of the resonances. In this case the existence of the universal smearing function is obvious.

The QCD situation is different and more complicated. There are condensates, so one would not expect the universality of the smearing function to be exact. Nevertheless, assuming (and testing) approximate universality may give a hint for choosing the relevant smearing function. To be more precise, let us assume the existence of the function $\Phi(s, \tilde{m}_b^2, \tilde{m}_u^2)$ with $\Psi(s) = \Phi(s, m_b^2, m_u^2)$ and $\Phi(s) = \Phi(s, \overline{m}_b^2, \overline{m}_u^2)$. Then combining (1) and (8) gives the relation

$$\int ds \rho(s, m_b^2, m_u^2)\Phi^2(s, m_b^2, m_u^2)$$

$$= \int ds \rho(s, \overline{m}_b^2, \overline{m}_u^2)\Phi^2(s, \overline{m}_b^2, \overline{m}_u^2)$$

$$- \overline{m}_b \langle \bar{q}q \rangle \Phi^2\left((\overline{m}_b + \overline{m}_u)^2, \overline{m}_b^2, \overline{m}_u^2\right). \tag{13}$$

If the condensate contribution is negligible, the constituent quark mass obtained from this equation is equal

to the current quark mass. If condensates give a large contribution, a priori it is not guaranteed that the solution for m_u exists at all. We shall see that for realistic values of the quark condensate the solution does exist and gives the value for the constituent quark m_u (or m_s) in the "expected" range.

3.1 The optimal smearing function

In view of the argument from the potential model discussed above, we may expect the existence of an "optimal" smearing function which is close to the wave function of the constituent quark model. In many applications of quark models the simple Ansatz

$$\Psi(s) \simeq \mathrm{e}^{-k^2/2\beta^2}, \quad k = \lambda^{1/2}(s, m_b^2, m_u^2)/2\sqrt{s}, \quad (14)$$

with $\beta \sim \Lambda_{\mathrm{QCD}}$, was found to give a good approximation to the exact solution of the spectral problem for ground-state mesons and to lead to a good description of the data. Thus we choose the trial function $\Phi(s, m_b^2, m_u^2)$ as follows:

$$\Phi(s, m_b^2, m_u^2) = \frac{m_b + m_u}{f_B M_B^2} \mathrm{e}^{-k^2(s, m_b^2, m_u^2)/2\beta_B^2}. \quad (15)$$

Replacing $m \to \overline{m}$ gives the smearing function $\Phi(s)$

$$\Phi(s) = \frac{\overline{m}_b + \overline{m}_u}{f_B M_B^2} \mathrm{e}^{-k^2(s, \overline{m}_b^2, \overline{m}_u^2)/2\beta_B^2}$$
$$\simeq \frac{\overline{m}_b}{f_B M_B^2} \mathrm{e}^{-(s-M_0^2)^2/4\overline{m}_b^2\beta^2}, \quad (16)$$

where $M_0 \simeq \overline{m}_b$ and in the last equation we used $\overline{m}_b^2 \gg \beta^2$. This relation prompts one to use the Gaussian smearing function for calculating f_B in (8) (i.e. to use the Gaussian sum rule and not the Borel one), and to place the center of the Gaussian near $\overline{m}_b^2$. Moreover, choosing $M_0 = \overline{m}_b$ improves the suppression of higher-dimension condensates: since $\Phi(s)$ has in this case a maximum at $s = \overline{m}_b^2$, the contribution of the dimension-4 condensate proportional to $\delta'(s - \overline{m}_b^2)$ vanishes, and thus higher-dimension condensates are suppressed by two powers of the inverse heavy-quark mass instead of one power as it occurs in Borel sum rules. So neglecting higher order condensates looks quite safe. In practice one can allow the difference $M_0 - \overline{m}_b$ to take a small non-zero value.

Before closing this subsection notice that in the case of mesons consisting of light quarks only one has $k^2 \simeq 4s$, so that for such hadrons (15) favors the use of Borel sum rules.

3.2 The constituent quark mass and the quark condensate

Using the Ansatz (15) we can rewrite (13) as follows:

$$\overline{m}_b \langle \bar{q}q \rangle = \int \mathrm{d}s \, \rho(s, \overline{m}_b^2, \overline{m}_u^2) \, \mathrm{e}^{-k^2(s, \overline{m}_b^2, \overline{m}_u^2)/\beta_B^2}$$

$$- \int \mathrm{d}s \, \rho(s, m_b^2, m_u^2) \, \mathrm{e}^{-k^2(s, m_b^2, m_u^2)/\beta_B^2}. \quad (17)$$

This relations gives a connection between the quark mass, the constituent mass and the condensate, but involves also m_b, $\overline{m}_b$, and the parameter β_B of the smearing function. To get rid of complications related to the presence of m_b we go to the limit $m_b \to \infty$. This procedure requires however some care: the spectral density $\rho(s, m_b^2, m_u^2)$ involves radiative corrections which so far have not been included into the consideration. As soon as the radiative corrections are included, it becomes crucial which precise definition of the heavy-quark mass is used. As known from the literature radiative corrections to the spectral density are big (almost 100%) for the pole heavy-quark mass [18], but they are less than 10% if one works with the $\overline{\mathrm{MS}}$ running mass [19]. Having in mind the small size of radiative corrections in terms of the $\overline{m}_b$, we require that the limit $m_b \to \infty$ is taken such that the ratio of the constituent m_b to the $\overline{\mathrm{MS}}$ mass $\overline{m}_b$ goes to unity,

$$\overline{m}_b/m_b \to 1. \quad (18)$$

Then the radiative corrections to (17) may be safely omitted to the accuracy we are interested in. Changing the integration variable, in the limit $m_b \to \infty$ (17) takes the form

$$\langle \bar{q}q \rangle \qquad (19)$$
$$= \frac{N_c}{\pi^2} \int_0^\infty \mathrm{d}k \, k^2 \, \mathrm{e}^{-k^2/\beta_\infty^2} \left\{ \frac{\overline{m}_u}{\sqrt{\overline{m}_u^2 + k^2}} - \frac{m_u}{\sqrt{m_u^2 + k^2}} \right\}.$$

Equation (19) is the first central result of this paper. The quark condensate and the current quark mass $\overline{m}_u$ depend on the scale, thus requiring a scale dependence of the quark-model parameters β_∞ and m_u. We shall discuss this dependence in the next section, and now we analyze the relation (19) at the chiral symmetry-breaking scale $\mu_\chi \simeq 1\,\mathrm{GeV}$.

First of all we note that below μ_χ the quark-model parameters $m_u(\mu)$ and $\beta_\infty(\mu)$ can be reasonably taken as constant values fixed by the specific potential model adopted. From the analysis of properties of the $Q\bar{q}$ mesons in [5] one expects the value $\beta_\infty = 0.6 \div 0.7\,\mathrm{GeV}$ in the heavy-quark limit. Neglecting the current mass of the light quark in numerical estimates, we find that the constituent quark mass $m_u = 220\,\mathrm{MeV}$ (obtained from the description of the meson transition form factors in [5] and prompted by the analysis of the meson mass spectrum in [24]) corresponds to the condensate $\langle \bar{q}q \rangle = -(230 \pm 15\,\mathrm{MeV})^3$. Interestingly, the condensate value does not vary much for m_u in the range $200 \div 350\,\mathrm{MeV}$. For instance for $m_u = 350\,\mathrm{MeV}$, $\langle \bar{q}q \rangle = -(260 \pm 15\,\mathrm{MeV})^3$. The error in this estimate results from the variation of β_∞ in the range $0.6 \div 0.7\,\mathrm{GeV}$.

The obtained condensate is negative in agreement with the Gell-Mann–Oakes–Renner relation [22] and compares favorably with the estimates $\langle \bar{q}q \rangle(1\,\mathrm{GeV}) = -(242 \pm 15\,\mathrm{MeV})^3$ [23].

A similar analysis can be done for the s quark. The parameter β_∞^s for the pseudoscalar $Q\bar{s}$ meson was found to

be only a few percent bigger than β_∞ [5,24]; the strange-quark condensate is $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle = 0.8 \pm 0.3$ [23]. Then allowing the range of values $\langle \bar{s}s \rangle = -(220 \pm 20 \, \text{MeV})^3$, $\beta_\infty^s = 0.6 \div 0.7 \, \text{GeV}$ and $\overline{m}_s = 110 \pm 10 \, \text{MeV}$ leads to a constituent mass of the strange quark in the range $m_s = 350 \pm 30 \, \text{MeV}$.

4 Running of the constituent quark mass and average momentum

Requiring the relation (19) to hold at any scale above μ_χ leads to the scale dependence of the quark-model parameters β_∞ and m_u.

In QCD the current quark mass and the quark condensate are multiplicatively renormalized in such a way that their product is renormalization group invariant (RGI), namely

$$\overline{m}_u(\mu) = \widehat{\overline{m}}_u \, Z_{\overline{m}}(\mu), \qquad (20)$$

$$\langle \bar{q}q \rangle(\mu) = \widehat{\langle \bar{q}q \rangle} \, Z_S(\mu), \qquad (21)$$

where $Z_S(\mu) = 1/Z_{\overline{m}}(\mu)$ at any renormalization scale μ, while $\widehat{\overline{m}}_u$ and $\widehat{\langle \bar{q}q \rangle}$ are RGI quantities. Within the $\overline{\text{MS}}$ scheme, at next-to-leading order (NLO) accuracy in the strong coupling constant $\alpha_s(\mu)$ one explicitly has [25]

$$Z_{\overline{m}}(\mu) = \left(\frac{\alpha_s(\mu)}{\pi} \right)^{4/\beta_0} \left[1 + \left(\frac{4\gamma_1}{\beta_0} - \frac{\beta_1}{\beta_0^2} \right) \frac{\alpha_s(\mu)}{\pi} \right], \quad (22)$$

where $\beta_0 = 11 - 2 \, n_f/3$, $\beta_1 = 102 - 38 \, n_f/3$ and $\gamma_1 = (101 - 10 \, n_f/3)/24$.

4.1 Running at high scales

Above the chiral scale the two quark-model parameters β_∞ and m_u run with the scale μ and their evolution is expected to be coupled in order to fulfill (19) and (21). Qualitatively, the constituent mass m_u decreases when the scale μ increases, while the parameter β_∞, which governs the average momentum of the constituent quark inside the hadron, increases with the scale μ. Thus, above a sufficiently high scale μ_0 ($> \mu_\chi$) the value of the parameter $\beta_\infty(\mu)$ becomes much larger than the value of the mass $m_u(\mu)$, so that in the RHS of (19) we can neglect both $\overline{m}_u^2$ and m_u^2 with respect to k^2. This leads to a simple expression for the quark condensate, namely

$$\langle \bar{q}q \rangle(\mu) \xrightarrow[\mu \geq \mu_0]{} \frac{N_c}{2\pi^2} \, \beta_\infty^2(\mu) \, [\overline{m}_u(\mu) - m_u(\mu)]. \qquad (23)$$

We have now to impose that the RHS of the above equation runs with the renormalization scale as in (21), using (20) for the scale dependence of the current quark mass. In general there is no unique solution. However, if we require that for $\mu \geq \mu_0$ the evolutions of $m_u(\mu)$ and $\beta_\infty(\mu)$ are decoupled, then there is a unique solution provided by

$$m_u(\mu) \xrightarrow[\mu \geq \mu_0]{} \widehat{m}_u \, Z_{\overline{m}}(\mu),$$

$$\beta_\infty(\mu) \xrightarrow[\mu \geq \mu_0]{} \widehat{\beta}_\infty / Z_{\overline{m}}(\mu), \qquad (24)$$

where $\widehat{m}_u$ and $\widehat{\beta}_\infty$ are RGI quantities satisfying the relation $\widehat{\langle \bar{q}q \rangle} = (N_c/2\pi^2) \, \widehat{\beta}_\infty^2 \, (\widehat{\overline{m}}_u - \widehat{m}_u)$. Note that the assumption of the same running for both the constituent and the current quark mass is quite natural and very plausible.

4.2 Running at intermediate scales $\mu_\chi \leq \mu \leq \mu_0$

For $\mu_\chi \leq \mu \leq \mu_0$ the scale dependences of $m_u(\mu)$ and $\beta_\infty(\mu)$ are coupled in order to fulfill (19) and (21). Requiring that the constituent mass has the same scale dependence as the current mass for $\mu \geq \mu_\chi$,

$$m_u(\mu) = m_u(\mu_\chi) \, Z_{\overline{m}}(\mu)/Z_{\overline{m}}(\mu_\chi), \qquad (25)$$

we obtain

$$\beta_\infty(\mu) = \beta_\infty(\mu_\chi) \, Z_\beta(\mu_\chi)/Z_\beta(\mu), \qquad (26)$$

where $m_u(\mu_\chi)$ and $\beta_\infty(\mu_\chi)$ are respectively the values of m_u and β_∞ up to the chiral scale μ_χ. From (24) one has that $Z_\beta(\mu) \simeq Z_{\overline{m}}(\mu)$ for $\mu \geq \mu_0$, while for
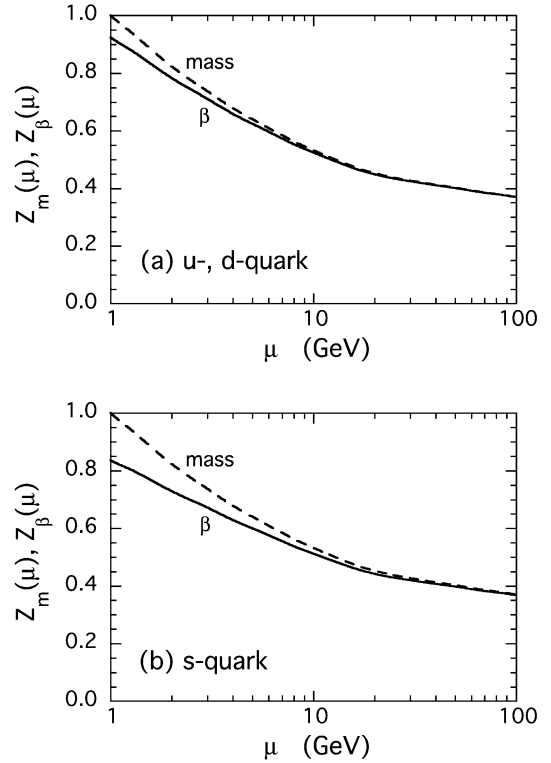

Fig. 2. The renormalization constants $Z_m(\mu) = Z_{\overline{m}}(\mu)$ (dashed lines) and $Z_\beta(\mu)$ (solid lines), divided by $Z_m(\mu = \mu_\chi = 1 \, \text{GeV})$, versus the renormalization scale μ. For the running of $Z_{\overline{m}}(\mu)$ we have considered (22) with $\alpha_s(M_Z) = 0.118$. The values of the constant $Z_\beta(\mu)$ are obtained as described in the text

$\mu_\chi \le \mu \le \mu_0$ the value of $Z_\beta(\mu)$ can be obtained numerically from (19) making use of (21). In Fig. 2 we report $Z_m(\mu) = Z_{\overline{m}}(\mu)$ given by (22) and $Z_\beta(\mu)$ obtained from (19) with $\overline{m}_u = 0$, $m_u(1\,\text{GeV}) = 250$ MeV and $\langle \bar{q}q \rangle(1\,\text{GeV}) = -(242\,\text{MeV})^3$ for u and d quark, and $\overline{m}_s(1\,\text{GeV}) = 110$ MeV, $m_s(1\,\text{GeV}) = 350$ MeV and $\langle \bar{s}s \rangle(1\,\text{GeV}) = -(220\,\text{MeV})^3$ for the s quark. Clearly, the scale dependence of $\beta_\infty(\mu)$ is quite close to the one of the mass in case of the light constituent u- and d quark, while larger differences appear only around the chiral scale μ_χ in case of the constituent s quark.

5 Conclusions and outlook

We have considered the relationship between QCD sum rules and the constituent quark model formulated in the form of spectral representations. Our main results are as follws.

(1) We compared the normalization condition for the wave function of a heavy-light pseudoscalar meson in constituent quark model with the QCD sum rule for the decay constant of the same pseudoscalar meson in QCD. We noticed that if one uses a specific version of QCD sum rules, in which the duality smearing function is close to the hadron wave function of the quark model, then effects related to condensates in QCD may be described in terms of the appearance of effective constituent quark masses.

(2) We gave arguments in favor of choosing the smearing functions of QCD sum rules close to the hadron wave functions of the constituent quark model. Applying sum rules for bound-state transition form factors in a confining potential model (a theory with confinement but without condensates), we have seen that the choice of the smearing functions equal to the bound-state wave functions leads to the exact result for the form factors. Although the condensates in QCD violate this exact relation, the approximate similarity of the smearing function with the wave function seems to remain a useful concept.

The knowledge of the hadron wave functions of the constituent quark model may then suggest the "optimal" choice of the smearing wave function of QCD sum rules, and thus the specific version of sum rules (Borel or Gaussian) to be used. For instance, the quark model wave functions speak in favor of using the Gaussian sum rules for B mesons and the Borel sum rules for mesons containing light quarks only.

(3) The similarity of the smearing and the wave functions allows us to obtain the relation between the quark condensate and the constituent quark mass. The constituent mass of the light quark $m_u = 220$ MeV corresponds to the quark condensate $\langle \bar{q}q \rangle = -(230 \pm 15\,\text{MeV})^3$ in a good agreement with the expected value of this quantity. Similarly a constituent mass of the strange quark equal to $m_s = 350$ MeV corresponds to a current quark mass $\overline{m}_s = 110 \pm 10$ MeV and a strange condensate in the range $\langle \bar{s}s \rangle = -(220 \pm 15\,\text{MeV})^3$.

(4) We addressed the problem of the scale dependence of our correspondence between constituent quark models and QCD sum rules. By imposing the known QCD running of the quark condensate we found explicitly the scale-dependence of the constituent quark mass and the average momentum of the quark inside the hadron. Our findings open the possibility to apply the constituent quark model above the scale of chiral symmetry breaking in a controlled way.

The observed correspondence between QCD sum rules and constituent quark models may have two important applications. First, it allows us to understand the parameters of the constituent quark model on the QCD basis and it also opens the possibility to apply such a model beyond the scale of chiral symmetry breaking. Second, it provides a physical motivation and control over the smearing functions in QCD sum rules. In spite of the obvious successes of the constituent quark model mentioned in the beginning of this paper, it is not easy to provide a reliable error estimate for its predictions. The context of QCD sum rules can give a firm theoretical basis for the quark model picture of hadrons.

The most interesting problem where the formulated ideas may be applied and tested is the physics of form factors. This work is in progress.

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